Algorithm 3.4 Marking immoralities in the construction of a perfect map

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    Procedure Mark-Immoralities (
      X = {X_1, ..., X_n},
      S    // Skeleton
      {U_{X_i,X_j} : 1 ≤ i, j ≤ n}    // Witnesses found by Build-PMap-Skeleton
    )
1     K ← S
2     for X_i, X_j, X_k such that X_i—X_j—X_k ∈ S and X_i—X_k ∉ S
3         // X_i—X_j—X_k is a potential immorality
4       if X_j ∉ U_{X_i,X_k} then
5         Add the orientations X_i → X_j and X_j ← X_k to K
6     return K
```

given Z. Nor will they be independent given a set $\boldsymbol{U}$ that contains Z. More precisely,

Proposition 3.1 *Let $\mathcal{G}^*$ be a P-map of a distribution P, and let X, Y and Z be variables that form an immorality $X \rightarrow Z \leftarrow Y$. Then, $P \not\models (X \perp Y \mid \boldsymbol{U})$ for any set $\boldsymbol{U}$ that contains Z.*

PROOF Let $\boldsymbol{U}$ be a set of variables that contains Z. Since Z is observed, the trail $X \rightarrow Z \leftarrow Y$ is active, and so X and Y are not d-separated in $\mathcal{G}^*$. Since $\mathcal{G}^*$ is a P-map of P, we have that $P^* \not\models (X \perp Y \mid \boldsymbol{U})$. ■

What happens in the complementary situation? Suppose $X—Z—Y$ in the skeleton, but is not an immorality. This means that one of the following three cases is in $\mathcal{G}^*$: $X \rightarrow Z \rightarrow Y$, $Y \rightarrow Z \rightarrow X$, or $X \leftarrow Z \rightarrow Y$. In all three cases, X and Y are d-separated only if Z is observed.

Proposition 3.2 *Let $\mathcal{G}^*$ be a P-map of a distribution P, and let the triplet X, Y, Z be a potential immorality in the skeleton of $\mathcal{G}^*$, such that $X \rightarrow Z \leftarrow Y$ is not in $\mathcal{G}^*$. If $\boldsymbol{U}$ is such that $P \models (X \perp Y \mid \boldsymbol{U})$, then $Z \in \boldsymbol{U}$.*

PROOF Consider all three configurations of the trail $X \rightleftharpoons Z \rightleftharpoons Y$. In all three, Z must be observed in order to block the trail. Since $\mathcal{G}^*$ is a P-map of P, we have that if $P \models (X \perp Y \mid \boldsymbol{U})$, then $Z \in \boldsymbol{U}$. ■

Combining these two results, we see that a potential immorality $X—Z—Y$ is an immorality if and only if Z is not in the witness set(s) for X and Y. That is, if $X—Z—Y$ is an immorality, then proposition 3.1 shows that Z is not in any witness set $\boldsymbol{U}$; conversely, if $X—Z—Y$ is not an immorality, the Z must be in every witness set $\boldsymbol{U}$. Thus, we can use the specific witness set $\boldsymbol{U}_{X,Y}$ that we recorded for X, Y in order to determine whether this triplet is an immorality or not: we simply check whether $Z \in \boldsymbol{U}_{X,Y}$. If $Z \notin \boldsymbol{U}_{X,Y}$, then we declare the triplet an immorality. Otherwise, we declare that it is not an immorality. The Mark-Immoralities procedure shown in algorithm 3.4 summarizes this process.