Algorithm 3.3 Recovering the undirected skeleton for a distribution P that has a P-map

Procedure Build-PMap-Skeleton (
  $\mathcal{X} = \{X_1, \ldots, X_n\}$, // Set of random variables
  P, // Distribution over $\mathcal{X}$
  d // Bound on witness set
)
1 Let $\mathcal{H}$ be the complete undirected graph over $\mathcal{X}$
2 **for** X_i, X_j in $\mathcal{X}$
3 $\boldsymbol{U}_{X_i,X_j} \leftarrow \emptyset$
4 **for** $\boldsymbol{U} \in$ Witnesses$(X_i, X_j, \mathcal{H}, d)$
5 // Consider $\boldsymbol{U}$ as a witness set for X_i, X_j
6 **if** $P \models (X_i \perp X_j \mid \boldsymbol{U})$ **then**
7 $\boldsymbol{U}_{X_i,X_j} \leftarrow \boldsymbol{U}$
8 Remove X_i—X_j from $\mathcal{H}$
9 **break**
10 **return** ($\mathcal{H}$,$\{\boldsymbol{U}_{X_i,X_j} : i, j \in \{1, \ldots, n\}\}$)

Witnesses$(X_i, X_j, \mathcal{H}, d)$ in line 4 specifies the set of possible witness sets that we consider for separating X_i and X_j. From our earlier discussion, if we assume a bound d on the indegree, then we can restrict attention to sets $\boldsymbol{U}$ of size at most d. Moreover, using the same analysis, we saw that we have a witness that consists either of the parents of X_i or of the parents of X_j. In the first case, we can restrict attention to sets $\boldsymbol{U} \subseteq \mathcal{X} - \{X_i, X_j\} - \mathrm{Nb}^{\mathcal{H}}_{X_i}$, where $\mathrm{Nb}^{\mathcal{H}}_{X_i}$ are the neighbors of X_i in the current graph $\mathcal{H}$; in the second, we can similarly restrict attention to sets $\boldsymbol{U} \subseteq \mathcal{X} - \{X_i, X_j\} - \mathrm{Nb}^{\mathcal{H}}_{X_j}$. Finally, we note that if $\boldsymbol{U}$ separates X_i and X_j, then also many of $\boldsymbol{U}$'s supersets will separate X_i and X_j. Thus, we search the set of possible witnesses in order of increasing size.

This algorithm will recover the correct skeleton given that $\mathcal{G}^*$ is a P-map of P and has bounded indegree d. If P does not have a P-map, then the algorithm can fail; see exercise 3.22. This algorithm has complexity of $O(n^{d+2})$ since we consider $O(n^2)$ pairs, and for each we perform $O((n-2)^d)$ independence tests. We greatly reduce the number of independence tests by ordering potential witnesses accordingly, and by aborting the inner loop once we find a witness for a pair (after line 9). However, for pairs of variables that are directly connected in the skeleton, we still need to evaluate all potential witnesses.

3.4.3.2 Identifying Immoralities

At this stage we have reconstructed the undirected skeleton S using Build-PMap-Skeleton. Now, we want to reconstruct edge direction. The main cue for learning about edge directions in $\mathcal{G}^*$ are immoralities. As shown in theorem 3.8, all DAGs in the equivalence class of $\mathcal{G}^*$ share the same set of immoralities. Thus, our goal is to consider *potential immoralities* in the skeleton and for each one determine whether it is indeed an immorality. A triplet of variables X, Z, Y is a *potential immorality* if the skeleton contains X—Z—Y but does not contain an edge between X and Y. If such a triplet is indeed an immorality in $\mathcal{G}^*$, then X and Y cannot be independent

potential immorality