Algorithm 13.4 Graph-cut algorithm for MAP in pairwise binary MRFs with submodular potentials

Procedure MinCut-MAP (
  ϵ // Singleton and pairwise submodular energy factors
)
  // Define the energy function
  for all i
    $\epsilon'_i \leftarrow \epsilon_i$
  Initialize $\epsilon'_{i,j}$ to 0 for all i, j
  for all pairs $i < j$
    $\epsilon'_i(1) \leftarrow \epsilon'_i(1) + (\epsilon_{i,j}(1,0) - \epsilon_{i,j}(0,0))$
    $\epsilon'_j(1) \leftarrow \epsilon'_j(1) + (\epsilon_{i,j}(1,1) - \epsilon_{i,j}(1,0))$
    $\epsilon'_{i,j}(0,1) \leftarrow \epsilon_{i,j}(1,0) + \epsilon_{i,j}(0,1) - \epsilon_{i,j}(0,0) - \epsilon_{i,j}(1,1)$

  // Construct the graph
  for all i
    if $\epsilon'_i(1) > \epsilon'_i(0)$ **then**
      $\mathcal{E} \leftarrow \mathcal{E} \cup \{(s, z_i)\}$
      cost$(s, z_i) \leftarrow \epsilon'_i(1) - \epsilon'_i(0)$
    else
      $\mathcal{E} \leftarrow \mathcal{E} \cup \{(z_i, t)\}$
      cost$(z_i, t) \leftarrow \epsilon'_i(0) - \epsilon'_i(1)$
    for all pairs $i < j$ such that $\epsilon'_{i,j}(0,1) > 0$
      $\mathcal{E} \leftarrow \mathcal{E} \cup \{(z_i, z_j)\}$
      cost$(z_i, z_j) \leftarrow \epsilon'_{i,j}(0,1)$

    $\boldsymbol{t} \leftarrow$ MinCut$(\{z_1, \ldots, z_n\}, \mathcal{E})$
      // MinCut returns $t_i = 1$ iff $z_i \in \mathcal{Z}_t$
    return $\boldsymbol{t}$

Because of submodularity, this term satisfies $\epsilon'_{i,j}(0,1) \geq 0$. The algorithm executes this transformation for every pairwise potential i, j. The resulting energy function can easily be converted into a graph using essentially the same construction that we used earlier; the only slight difference is that for our new energy function $\epsilon'_{i,j}(v_i, v_j)$ we need to introduce only the edge (z_i, z_j), with cost $\epsilon'_{i,j}(0,1)$; we do not introduce the opposite edge (z_j, z_i). We now use the same mapping between s-t cuts in the graph and assignment to the variables $X_1, \ldots, X_n$. It is not difficult to verify that the cost of an s-t cut $\mathcal{C}$ in the resulting graph is precisely $E(\xi^{\mathcal{C}}) + \text{Const}$ (see exercise 13.14). Thus, finding the minimum cut in this graph directly gives us the cost-minimizing assignment ξ^{map}.

Note that for pairwise submodular energy, there is an LP relaxation of the MAP integer optimization, which is tight. Thus, this result provides another example where having a tight LP relaxation allows us to find the optimal MAP assignment.