Algorithm 13.3 Calibration using max-product BP in a Bethe-structured cluster graph

Procedure Generalized-MP-BP (
  Φ, // Set of factors
  $\mathbf{R}$, // Set of regions
  $\{\kappa_r\}_{r\in\mathbf{R}}$, $\{\kappa_i\}_{X_i\in\mathcal{X}}$ // Counting numbers
)
  $\rho_i \leftarrow 1/\kappa_i$
  $\rho_r \leftarrow 1/\kappa_r$
  Initialize-CGraph
  while region graph is not max-calibrated
    Select $\boldsymbol{C}_r$ and $X_i \in \boldsymbol{C}_r$
      $\delta_{i\to r}(X_i) \leftarrow \left[\left(\prod_{r'\neq r}\delta_{i\to r'}(X_i)\right)^{\rho_i}\left(\max_{\boldsymbol{C}_r - X_i}\psi_r(\boldsymbol{C}_r)\left(\prod_{X_j\in\boldsymbol{C}_r, j\neq i}\delta_{j\to r}\right)^{\rho_r}\right)\right]^{-\frac{1}{\rho_i+\rho_r}}$
    for each region $r \in \mathbf{R} \cup \{1,\ldots,n\}$
      $\beta_r(\boldsymbol{C}_r) \leftarrow \left(\psi_r(\boldsymbol{C}_r)\prod_{X_i\in\boldsymbol{C}_r}\delta_{i\to r}(X_i)\right)^{\rho_r}$
  return $\{\beta_r\}_{r\in\mathbf{R}}$

13.4.2.1 Max-Product with Counting Numbers

We begin with a reminder of the notion of belief propagation with counting numbers. For concreteness, we also provide the max-product variant of a message passing algorithm for this case, although (as we mentioned) the max-product variant can be obtained from the sum-product algorithm using a simple syntactic substitution.

counting numbers

In section 11.3.7, we defined a set of sum-product message passing algorithms; these algorithms were defined in terms of a set of *counting numbers* that specify the extent to which entropy terms for different subsets of variables are counted in the entropy approximation used in the energy functional. For a given set of counting numbers, one can derive a message passing algorithm by using the fixed point equations obtained by differentiating the Lagrangian for the energy functional, with its sum-product calibration constraints. The standard belief propagation algorithm is obtained from the Bethe energy approximation; other sets of counting numbers give rise to other message passing algorithms.

Bethe cluster graphs

As we discussed, one can take these sum-product message passing algorithms (for example, those in exercise 11.17 and exercise 11.19) and convert them to produce a max-product variant by simply replacing each summation operation as maximization. For concreteness, in algorithm 13.3, we repeat the algorithm of exercise 11.17, instantiated to the max-product setting. Recall that this algorithm applies only to *Bethe cluster graphs*, that is, graphs that have two levels of regions: "large" regions r containing multiple variables with counting numbers κ_r, and singleton regions containing individual variables X_i with counting numbers κ_i; all factors in Φ are assigned only to large regions, so that $\psi_i = 1$ for all i.