Algorithm 13.1 Variable elimination algorithm for MAP. The algorithm can be used both in its max-product form, as shown, or in its max-sum form, replacing factor product with factor addition.

Procedure Max-Product-VE (
  Φ, // Set of factors over $\boldsymbol{X}$
  $\prec$ // Ordering on $\boldsymbol{X}$
)
Let $X_1, \ldots, X_k$ be an ordering of $\boldsymbol{X}$ such that
  $X_i \prec X_j$ iff $i < j$
  for $i = 1, \ldots, k$
    $(\Phi, \phi_{X_i}) \leftarrow$ Max-Product-Eliminate-Var(Φ, X_i)
  $\boldsymbol{x}^* \leftarrow$ Traceback-MAP$(\{\phi_{X_i} : i = 1, \ldots, k\})$
  return $\boldsymbol{x}^*, \Phi$ // Φ contains the probability of the MAP

Procedure Max-Product-Eliminate-Var (
  Φ, // Set of factors
  Z // Variable to be eliminated
)
  $\Phi' \leftarrow \{\phi \in \Phi : Z \in \mathit{Scope}[\phi]\}$
  $\Phi'' \leftarrow \Phi - \Phi'$
  $\psi \leftarrow \prod_{\phi \in \Phi'} \phi$
  $\tau \leftarrow \max_Z \psi$
  return $(\Phi'' \cup \{\tau\}, \psi)$

Procedure Traceback-MAP (
  $\{\phi_{X_i} : i = 1, \ldots, k\}$
)
  for $i = k, \ldots, 1$
    $\boldsymbol{u}_i \leftarrow (x^*_{i+1}, \ldots, x^*_k)\langle \mathit{Scope}[\phi_{X_i}] - \{X_i\}\rangle$
      // The maximizing assignment to the variables eliminated after X_i
    $x^*_i \leftarrow \arg\max_{x_i} \phi_{X_i}(x_i, \boldsymbol{u}_i)$
      // x^*_i is chosen so as to maximize the corresponding entry in the factor, relative to the previous choices $\boldsymbol{u}_i$
  return $\boldsymbol{x}^*$

As we have discussed, the result of the computation is a max-marginal $\mathit{MaxMarg}_{\tilde{P}_\Phi}(X_i)$ over the final uneliminated variable, X_i. We can now choose the maximizing value x^*_i for X_i. Importantly, from the definition of max-marginals, we are guaranteed that there exists some assignment ξ^* consistent with x^*_i. But how do we construct such an assignment?

We return once again to our simple example:

Example 13.3 *Consider the network of example 13.1, but now assume that we wish to find the actual assignment* $a^*, b^* = \arg\max_{A,B} P(A, B)$. *As we discussed, we first compute the internal maximization*