
Algorithm 11.3 Algorithm to construct a saturated region graph

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Procedure Build-Saturated-Region-Graph (
    R // a set of initial regions
)
1  repeat
2    For any  $r_1, r_2 \in \mathbf{R}$ 
3       $\mathbf{Z} \leftarrow \text{Scope}[r_1] \cap \text{Scope}[r_2]$ 
4      if  $\mathbf{Z} \neq \emptyset$ 
5        and R does not contain a region with scope  $\mathbf{Z}$  then
6          create region  $r$  with  $\text{Scope}[r] = \mathbf{Z}$ 
7           $\mathbf{R} \leftarrow \mathbf{R} \cup \{r\}$ 
8  Until convergence
9  Initialize  $\mathcal{R}$  as an empty graph with R as vertices
10 for each  $r_1 \neq r_2 \in \mathbf{R}$ 
11   if  $\text{Scope}[r_2] \subset \text{Scope}[r_1]$  and
12    not exist  $r_3 \in \mathbf{R}$  such that  $\text{Scope}[r_2] \subset \text{Scope}[r_3] \subset \text{Scope}[r_1]$  then
13      add an arc  $r_1 \rightarrow r_2$  to the region graph  $\mathcal{R}$ 
14 return  $\mathcal{R}$ 

```

We start with initial set of regions. Often, these regions will be the initial factors in P_Φ , although we can decide to work with bigger regions that capture some more global interactions. We then extend this set of regions into a valid region graph, where our goal is to represent appropriately any subset of variables that is shared by some of the regions. We therefore expand the set of regions to be closed under intersections. We connect these regions so that the upward closure of each region contains all of its supersets. The full procedure is shown in algorithm 11.3. Unlike the Bethe approximation, this region graph maintains the consistency of higher-order marginals. The example of figure 11.9 is an example of running this procedure on the original set of regions $\{A, B, C\}$, $\{B, C, D\}$, and $\{A, C, D\}$. As our previous discussion suggests, this procedure guarantees a region graph that satisfies the region graph condition.

Belief Propagation in Region Graphs Given a region graph, we are faced with the task of optimizing the free energy associated with its structure: