Algorithm 23.2 Generalized variable elimination for joint factors in influence diagrams

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Procedure Generalized-VE-for-IDs (
   Φ,   // Set of joint (probability,utility) factors
   W_1, ..., W_k   // List of variables to be eliminated
)
  for i = 1, ..., k
    Φ' ← {φ ∈ Φ : W_i ∈ Scope[φ]}
    ψ ← ⊕_{φ∈Φ'} φ
    τ ← marg_{W_i}(ψ)
    Φ ← Φ − Φ' ∪ {τ}
  φ* ← ⊕_{φ∈Φ} φ
  return φ*
```

factor for each decision variable. Recall that our expected utility is defined as:

$$\mathrm{EU}[\mathcal{I}[\sigma]] = \sum_{W\in\mathcal{X}\cup\mathcal{D}} \prod_{W\in\mathcal{X}\cup\mathcal{D}} \phi_W \cdot (\sum_{V\in\mathcal{U}} \mu_V).$$

Let γ^* be the marginalization over all variables of the combination of all of the joint factors:

$$\gamma^* = (\phi^*, \mu^*) = \mathit{marg}_{\emptyset}(\bigoplus_{(W\in\mathcal{X}\cup\mathcal{U})} [\gamma_W]). \tag{23.8}$$

Note that the factor has empty scope and is therefore simply a pair of numbers. We can now show the following simple result:

Proposition 23.1 *For γ^* defined in equation (23.8), we have: $\gamma^* = (1, \mathrm{EU}[\mathcal{I}[\sigma]])$.*

The proof follows directly from the definitions and is left as an exercise (exercise 23.2).

Of course, as we discussed, we want to interleave the marginalization and combination steps. An algorithm implementing this idea is shown in algorithm 23.2. The algorithm returns a single joint factor (ϕ, μ).

Example 23.16 *Let us consider the behavior of this algorithm on the influence diagram of example 23.12, assuming again that we have a decision rule for D, so that we have only chance variables and utility variables. Thus, we initially have five joint factors derived from the probability factors for A, B, C, D, E; for example, we have $\gamma_B = (P(B \mid A), \mathbf{0}_{A,B})$. We have two joint factors γ_1, γ_2 derived from the utility variables V_1, V_2; for example, we have $\gamma_2 = (\mathbf{1}_{C,E}, V_2(C, E))$.*

Now, consider running our generalized variable elimination algorithm, using the elimination ordering C, A, E, B, D. Eliminating C, we first combine γ_C, γ_2 to obtain:

$$\gamma_C \bigoplus \gamma_2 = (P(C), V_2(C, E)),$$

where the scope of both components is taken to be C, E. We then marginalize C to obtain:

$$\begin{aligned} \gamma_3(E) &= \left(\mathbf{1}_E, \frac{\sum_C (P(C)V_2(C, E))}{\mathbf{1}_E}\right) \\ &= (\mathbf{1}_E, \boldsymbol{E}_{P(C)}[V_2(C, E)]). \end{aligned}$$